The importance of Weber-Maxwell electrodynamics in electrical engineering

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Weber-Maxwell electrodynamics combines classical Weber electrodynamics and Maxwell's equations, including all four field equations and the Lorentz force, into a single de facto equivalent three-dimensional wave equation. From classical Weber electrodynamics, Weber-Maxwell electrodynamics inherits properties in which the concept of the magnetic field is unnecessary, and Newton's third law is satisfied under all circumstances. From Maxwell's electrodynamics, Weber-Maxwell electrodynamics inherits the ability to be compatible with electromagnetic waves. This article shows that in Weber-Maxwell electrodynamics, all conservation laws are satisfied, and that electromagnetic waves in an isolated system do not possess energy and momentum, but only mediate them between particles of matter. Furthermore, the article shows that the modern formulation of Weber electrodynamics is clearly superior to standard electrodynamics in electrical engineering, because it not only eliminates the internal contradictions, but also represents considerable simplification and compression.

***Index Terms*—Vector wave equation, Electromagnetic forces, Electromagnetic propagation, Weber electrodynamics, Computational electromagnetics**

I. Introduction

Weber electrodynamics is a theory of electromagnetism from before the time when Maxwell's equations and special relativity were developed; it dates back to the works of A.-M. Ampère, W. Weber and C. F. Gauss in the middle of the 19th century [1]–[3]. Weber electrodynamics is now considered obsolete, although it has some remarkable and highly attractive features that are absent in Lorentz-Einstein electrodynamics.

The most important of these features is that, in Weber electrodynamics, the concepts of the magnetic field[1], are obsolete, because Weber electrodynamics generalizes the electrostatic Coulomb force to a velocity-dependent electrodynamic force. A highly important and essential aspect of the Weber force is that, despite this modification, it remains a central force that satisfies Newton's third law.

In contrast, the Liénard-Schwarzschild force [4, Eq. (26)], which follows from the canonical application of Maxwell's equations and the Lorentz force for two point charges, violates Newton's third law, even at non-relativistic velocities. Consequently, at everyday speeds, a formal violation of the law of conservation of momentum occurs. J. P. Wesley, for instance, has addressed this subject [5]:

> It should be remarked that a failure to obey Newton's third law is a very serious matter; as it implies drastic consequences, such as the violation of the conservation of energy, the ability to propel a space craft using only forces internal to the space craft itself, and the ability to lift oneself by one's own boot straps.

[1] magnetic flux density $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$ and Lorentz force

Later, J. P. Wesley showed that Lorentz-Einstein electrodynamics allows for absurd conclusions, even for standard engineering problems. Slightly sarcastically, he writes:

> Depending upon how one chooses portions 1 and 2, one can obtain a nonvanishing force with any value at all (within limits). Such a loop would be very convenient to drive an automobile or propel a space ship. One could obtain the desired magnitude of the force without having to change anything physically; one need only alter the mathematical labels.

Experienced electrical engineers know that Lorentz-Einstein electrodynamics must be used with caution, and they instinctively know the limits of the theory. However, from a scientific point of view, this situation is unsatisfactory, especially because Lorentz-Einstein electrodynamics repeatedly tempts inexperienced practitioners to think about perpetual motion machines or to research propulsion devices that violate the conservation of momentum.

Physicists have also long studied the intrinsic contradictions of Lorentz-Einstein electrodynamics. For example, the missing momentum has often been argued to be carried by the radiation field [6]. However, good arguments in the literature have shown that this explanation is not sufficient [7]–[9]. Furthermore, experiments have investigated these aspects and demonstrated that, in the low energy domain, Weber electrodynamics not only explains the behavior of nature in a simpler and more satisfactory manner, but also provides better predictions [10]–[14].

Unfortunately, Weber electrodynamics has two critical drawbacks that severely limit its practical value in electrical engineering and modern physics. One is the lack of connection with Maxwell's equations and the absence of a field concept, thus disqualifying Weber electrodynamics from use in all fields of electrical engineering involving electromagnetic waves and their propagation. In addition, classical Weber electrodynamics cannot be applied at relativistic velocities, thereby rendering it largely useless for particle physics, although it would actually be predestined for work with point charges. In addition, its value in atomic and plasma physics is low, although some interesting approaches have been reported [15]–[18].

The two aforementioned drawbacks are substantial and are clearly the reason why Weber electrodynamics, despite its elegant concept and superior properties, never became accepted in the low-energy domain and nearly fell into obscurity. However, the first drawback exists only if Weber electrodynamics is used in its original form from the middle of the 19th century, i.e., before the introduction of Maxwell's equations. Indeed, Weber electrodynamics can also be generalized to Weber-Maxwell electrodynamics and can be formulated as a field theory in which Maxwell's equations are compressed and summarized into a single wave equation [19]. That this is possible demonstrates that Weber electrodynamics is as closely related to Maxwell's equations, as is canonical Lorentz-Einstein electrodynamics. Furthermore, the fusion of Weber electrodynamics with Maxwell's equations eliminates one of the two fundamental disadvantages of Weber electrodynamics.

This article is primarily aimed at showing that the conservation laws hold in Weber-Maxwell electrodynamics. In particular, in Weber-Maxwell electrodynamics, the radiation field is demonstrated to have neither energy nor momentum, and the electromagnetic field serves only as a mediator. Why this finding does not represent an experimental contradiction is also explained.

In conclusion, Weber-Maxwell electrodynamics is shown to be a theory that is clearly superior to standard electrodynamics in the non-relativistic domain, because the concept of the magnetic field is no longer necessary, and the conservation laws are guaranteed to be valid even in the presence of electromagnetic waves. Moreover, from a purely practical point of view, Weber-Maxwell electrodynamics has considerable advantages because it allows engineering problems to be solved quickly and easily through numerically integrating the solutions for point charges or point antennas along arbitrary conductor paths or antenna structures[2]. Therefore, Weber-Maxwell electrodynamics possesses all the properties expected of a good theory in electrical engineering.

II. Modern formulation of Weber electrodynamics

In contrast to standard electrodynamics, Weber electrodynamics is a theory describing the force between two moving point charges. Terms such as current density, charge density, and magnetic or electric field strength do not yet appear at this level of abstraction. Instead, a formula is given for the electromagnetic force between a force-emitting point charge and a force-absorbing point charge. However, the original formulation of Weber electrodynamics is not yet based on a differential equation; consequently, only oscillations, but no wave phenomena, can be represented by classical Weber electrodynamics.

[2]With Lorentz-Einstein electrodynamics, this is not possible, because the Liénard-Schwarzschild force does not represent a reasonable solution for the electromagnetic field of the force of a point charge

However, classical Weber electrodynamics can be combined with Maxwell's equations and expressed with the wave equation [19]

$$\boxed{\Box \boldsymbol{F}_{ds} = -\frac{q_d\, q_s}{\varepsilon_0}\left(\frac{\partial}{\partial t}\,\frac{\dot{\boldsymbol{r}}_{ds}}{c^2}\;\gamma(\dot{\boldsymbol{r}}_{ds})\,\delta(\boldsymbol{r}_{ds}) + \gamma(\dot{\boldsymbol{r}}_{ds})\,\nabla\,\delta(\boldsymbol{r}_{ds})\right).} \tag{1}$$

In words, the differential equation (1) describes the relation of the force $\boldsymbol{F}_{ds}$ exerted by the point charge q_s on another point charge q_d by expressing how the force depends on the generally time-varying distance vector $\boldsymbol{r}_{ds}$. The Lorentz factor is defined as usual by:

$$\gamma(\boldsymbol{v}) := \frac{1}{\sqrt{1-\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}}} \tag{2}$$

δ is the Dirac delta function. The vector $\boldsymbol{r}_{ds}$ represents the difference or distance vector of the two trajectories $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$:

$$\boldsymbol{r}_{ds} := \boldsymbol{r}_d - \boldsymbol{r}_s. \tag{3}$$

The time derivative of the difference vector of the two trajectories is consequently the difference velocity or relative velocity:

$$\dot{\boldsymbol{r}}_{ds} = \dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}_s. \tag{4}$$

The wave equation (1) summarizes all four Maxwell equations and the Lorentz force. In the form (1), it differs from the wave equation of Lorentz-Einstein electrodynamics in the rest frame of the receiver by only an additional Lorentz factor. However, this factor has no effect on the time behavior of the electromagnetic waves. The purpose of the additional Lorentz factor is only to re-normalize the force such that the concept of the magnetic field becomes obsolete. Compared with canonical electrodynamics, with its four-dimensional electromagnetic field tensor, this equation represents an immense simplification, which should not be underestimated.

From the wave equation (1), both of Einstein's postulates are clearly fulfilled, because the force $\boldsymbol{F}_{ds}$ depends on only the difference vector $\boldsymbol{r}_{ds}$ and the difference velocity $\dot{\boldsymbol{r}}_{ds}$. Both are relative quantities, and the velocity of an observer does not enter the wave equation. Consequently, the principle of relativity is concluded to be satisfied, and the force has exactly the same value in each inertial frame. Furthermore, the d'Alembert operator shows that the force between the point charges q_s and q_d propagates with velocity c independently of their relative velocity $\dot{\boldsymbol{r}}_{ds}$.

For approximately uniformly moving point charges, the wave equation (1) has the solution

$$\boldsymbol{F}_{ds} = \boldsymbol{F}_W(q_d, q_s, \boldsymbol{r}_{ds}, \dot{\boldsymbol{r}}_{ds}) \tag{5}$$

with

$$\boxed{\boldsymbol{F}_W(q_d, q_s, \boldsymbol{r}, \boldsymbol{v}) = \frac{\gamma(\boldsymbol{v})^2}{\left(1+\left(\frac{\gamma(\boldsymbol{v})}{c}\,\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{\|\boldsymbol{r}\|}\right)^2\right)^{\frac{3}{2}}}\,\boldsymbol{F}_c(q_d, q_s, \boldsymbol{r})} \tag{6}$$

where $\boldsymbol{F}_c$ is the Coulomb force

$$\boldsymbol{F}_c(q_d, q_s, \boldsymbol{r}) = \frac{q_d\, q_s}{4\,\pi\,\varepsilon_0}\,\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|^3}. \tag{7}$$

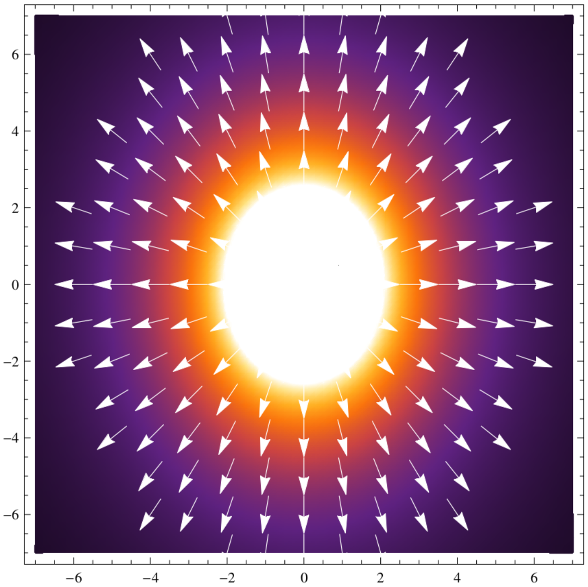

Fig. 1. The electromagnetic force of a point charge located at the coordinate origin is shown from the perspective of a test charge moving with relative velocity $\boldsymbol{v} = c/2\,\boldsymbol{e}_x$. As indicated, the force is elliptically deformed with respect to the Coulomb force, but it is nonetheless a central force.

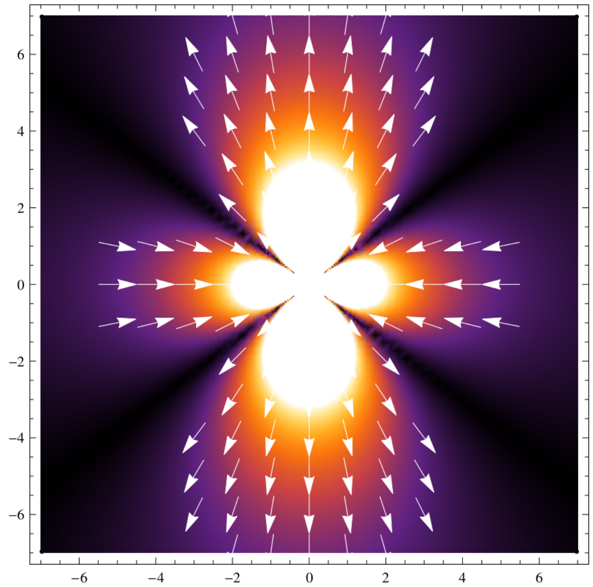

Fig. 2. The same situation as in Figure 1 is shown, but all force components not depending on the velocity (Coulomb force) have been removed. The residual shown is the equivalent of the magnetic force in Weber electrodynamics.

For $\|v\| \ll c$, $\boldsymbol{F}_W$ becomes the classical Weber force (24), as will be shown below, and for $\|\boldsymbol{v}\| = 0$, the Coulomb force is obtained.

For illustration, Figure 1 shows the shape of the force between two point charges at high relative velocities. As indicated, the force is elliptically deformed but nevertheless remains a central force. Figure 2 shows the force component that results when the Coulomb force is removed from the force in Figure 1. Because the Coulomb force has no time dependence, the remainder is the velocity-dependent part and hence the component corresponding to the magnetic force in Weber electrodynamics. However, splitting into magnetic and electric components is not usually performed in Weber electrodynamics, because doing so is unnecessary.

However, the wave equation (1) not only provides the classical Weber force as a solution but also enables analysis of considerably more complex problems. In particular, the electromagnetic field $\boldsymbol{F}_{ds}$ of a point antenna, which may also move with high relative velocity, can be calculated. To obtain the field in the usual field notion, $\boldsymbol{r}_d$ and $\dot{\boldsymbol{r}}_d$ are left open as parameters. The calculated solutions are consistent with the solutions in textbooks, provided that the point antenna and receiver are at rest relative to each other. For a fast moving point antenna, solutions are automatically obtained that clearly indicate that the force between the point charges propagates at the speed of light. The Lorentz transformation is not required. In a later section, the field of the point antenna is discussed in more detail.

As can easily be seen, the force (6) satisfies Newton's third law $\boldsymbol{F}_W(q_d, q_s, \boldsymbol{r}, \boldsymbol{v}) = -\boldsymbol{F}_W(q_s, q_d, -\boldsymbol{r}, -\boldsymbol{v})$, because when the source and the receiver are swapped, the force only changes its sign. Critically, this aspect is true even in general, i.e., when electromagnetic waves are present, or when the two point charges are far from each other, so that the finite propagation velocity of the force is relevant. This can be proven by exchanging the source and recipient of the force in the wave equation (1), because with aid of the relations $\nabla\,\delta(\boldsymbol{r}_{ds}) = -\nabla\,\delta(\boldsymbol{r}_{sd})$, $\delta(\boldsymbol{r}_{ds}) = \delta(\boldsymbol{r}_{sd})$, $\gamma(\dot{\boldsymbol{r}}_{ds}) = \gamma(\dot{\boldsymbol{r}}_{sd})$ and $\dot{\boldsymbol{r}}_{ds} = -\dot{\boldsymbol{r}}_{sd}$, it follows that

$$\Box\,\boldsymbol{F}_{sd} = \frac{q_d\,q_s}{\varepsilon_0}\left(\frac{\partial}{\partial t}\,\frac{\dot{\boldsymbol{r}}_{sd}}{c^2}\,\gamma(\dot{\boldsymbol{r}}_{sd})\,\delta(\boldsymbol{r}_{sd}) + \gamma(\dot{\boldsymbol{r}}_{sd})\,\nabla\,\delta(\boldsymbol{r}_{sd})\right) \tag{8}$$

and therefore

$$\boxed{\boldsymbol{F}_{ds} = -\boldsymbol{F}_{sd}.} \tag{9}$$

That Newton's third law (9) is satisfied even in the presence of electromagnetic waves is highly important for the validity of the conservation laws. In particular, it shows that in Weber-Maxwell electrodynamics, electromagnetic waves have no energy and no momentum, but mediate both. Thus, Weber-Maxwell electrodynamics is strongly distinguished on an interpretive basis from Lorentz-Einstein electrodynamics, wherein energy and momentum must be assigned to the waves themselves. From an experimental point of view, however, no problem exists, because an electromagnetic wave can cause a measurable change in energy and momentum at the receiver in Weber-Maxwell electrodynamics. This is true even if the wave might have traveled a very long distance. Importantly, in Weber-Maxwell electrodynamics, the total energy and total momentum of all particles in the universe are conserved quantities without the electromagnetic field. However, if only a small part of the system, consisting only of the receiver and the incident electromagnetic wave without the transmitter, is considered, one must of course also assign energy and momentum to the wave, because the system without the transmitter does not represent an isolated system.

III. Proof of the conservation laws

The following section formally proves that in Weber-Maxwell electrodynamics in isolated systems, energy, momentum and angular momentum are conserved quantities. This proof is presented for several reasons. The first reason is to show that J. C. Maxwell and H. Helmholtz [20, p. 396-397] were incorrect in their conjecture that Weber electrodynamics violates the conservation of energy. The second reason is to clarify that conservative forces may also depend on the relative velocity and relative acceleration, and that the concept of potential energy can easily be applied to such generalized conservative forces. Finally, the requirements to guarantee compliance with the conservation laws are identified. This aspect is important, because whether and how Weber-Maxwell electrodynamics is suitable for the relativistic domain, and how the formulas for kinetic energy and momentum must be adapted are not yet clear.

A. Potential energy in Weber electrodynamics

We start with the total energy E of a two-particle system:

$$E = T_d + T_s + U_{ds}, \tag{10}$$

where $U_{ds} = U_{sd}$ is the potential energy possessed by the point charge q_d in the field of the point charge q_s. T_s and T_d are the kinetic energies

$$T_d = \frac{1}{2}\, m_d\, \|\dot{\boldsymbol{r}}_d\|^2, \quad T_s = \frac{1}{2}\, m_s\, \|\dot{\boldsymbol{r}}_s\|^2 \tag{11}$$

of the two point charges. Of note, for the kinetic energy, the differential velocity $\dot{\boldsymbol{r}}_{ds}$ is not used; instead, the velocities from the perspective of a laboratory system are used. Consequently, the kinetic energies as well as the total energy E usually have different values in each frame of reference. This aspect is not a property of Weber electrodynamics but is a basic characteristic of Newtonian mechanics.

If the energy E is to be a conserved quantity, then the time derivative of equation (10) must disappear. For this reason, we insert equation (11) and calculate the derivative. In this way, we obtain the following equation

$$\dot{E} = m_d\, \dot{\boldsymbol{r}}_d \cdot \ddot{\boldsymbol{r}}_d + m_s\, \dot{\boldsymbol{r}}_s \cdot \ddot{\boldsymbol{r}}_s + \dot{U}_{ds} = 0. \tag{12}$$

A comparison with Newton's second law

$$\boldsymbol{F}_{ds} = m_d\, \ddot{\boldsymbol{r}}_d, \quad \boldsymbol{F}_{sd} = m_s\, \ddot{\boldsymbol{r}}_s \tag{13}$$

shows that equation (12) is equivalent to

$$\boldsymbol{F}_{ds} \cdot \dot{\boldsymbol{r}}_d + \boldsymbol{F}_{sd} \cdot \dot{\boldsymbol{r}}_s + \dot{U}_{ds} = 0. \tag{14}$$

Because in Weber-Maxwell electrodynamics, Newton's third law (9) is always valid, it follows that

$$\boldsymbol{F}_{ds} \cdot (\dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}_s) + \dot{U}_{ds} = 0, \tag{15}$$

which in turn can be rewritten as

$$\boldsymbol{F}_{ds} \cdot \dot{\boldsymbol{r}}_{ds} + \dot{U}_{ds} = 0 \tag{16}$$

because of the definition (3). This equation is fulfilled if the relation

$$\boldsymbol{F}_{ds} = -\frac{\boldsymbol{r}_{ds}}{\boldsymbol{r}_{ds} \cdot \dot{\boldsymbol{r}}_{ds}}\, \dot{U}_{ds} \tag{17}$$

is valid, as can be easily verified by inserting it into equation (16).

Therefore, the conservation of energy in the two-particle system is guaranteed if the force (5) can be expressed with equation (17) based on potential energy U_{ds}. In fact, such a formula exists:

$$U_{ds} = U_W(q_d, q_s, \boldsymbol{r}_{ds}, \dot{\boldsymbol{r}}_{ds}) \tag{18}$$

with

$$\boxed{U_W(q_d, q_s, \boldsymbol{r}, \boldsymbol{v}) = \frac{U_c(q_d, q_s, \boldsymbol{r})}{\sqrt{1 + \left(\frac{\gamma(\boldsymbol{v})}{c}\, \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{\|\boldsymbol{r}\|}\right)^2}}} \tag{19}$$

where

$$U_c(q_d, q_s, \boldsymbol{r}) := \frac{q_d\, q_s}{4\,\pi\,\varepsilon_0}\, \frac{1}{\|\boldsymbol{r}\|} \tag{20}$$

is the potential energy of a point charge q_d at rest in the field of another resting point charge q_s. Notably, in Weber-Maxwell electrodynamics, potential energy and the scalar potential must be clearly distinguished.

That equation (18) together with equation (19) in fact provide a valid definition of the potential energy can be verified through substitution into equation (17) and calculation of the derivative. This process gives the force

$$\boldsymbol{F}_{ds} = \frac{\left(\kappa + \gamma(\dot{\boldsymbol{r}}_{ds})^2\right)\, \boldsymbol{F}_c(q_d, q_s, \boldsymbol{r}_{ds})}{\left(1 + \left(\frac{\gamma(\dot{\boldsymbol{r}}_{ds})}{c}\, \frac{\boldsymbol{r}_{ds}\cdot\dot{\boldsymbol{r}}_{ds}}{\|\boldsymbol{r}_{ds}\|}\right)^2\right)^{\frac{3}{2}}}, \tag{21}$$

with

$$\kappa := \frac{\gamma(\dot{\boldsymbol{r}}_{ds})^4}{c^4}\, (\ddot{\boldsymbol{r}}_{ds} \cdot \boldsymbol{r}_{ds})\, (\boldsymbol{r}_{ds} \cdot \dot{\boldsymbol{r}}_{ds}) + \frac{\gamma(\dot{\boldsymbol{r}}_{ds})^2}{c^2}\, \ddot{\boldsymbol{r}}_{ds} \cdot \boldsymbol{r}_{ds} \tag{22}$$

which for $\ddot{\boldsymbol{r}}_{ds} \approx \boldsymbol{0}$ corresponds to the solution [19, Eq. (54)] of the wave equation for uniformly moving point charges (6).

Furthermore, the force (21) is for small relative velocities $\|\dot{\boldsymbol{r}}_{ds}\| \ll c$ equivalent to the Weber force (24). To provide a demonstration, the substitution $\dot{\boldsymbol{r}}_{ds} \to v\, \dot{\boldsymbol{r}}_{ds}$ is performed, and the resulting equation with respect to the auxiliary scalar variable v is developed into a Taylor series of second order. If $v = 1$ is set, the following is obtained

$$\begin{aligned} \boldsymbol{F}_{ds} \;&= 1 + \frac{1}{c^2}\left(\|\dot{\boldsymbol{r}}_{ds}\|^2 - \frac{3}{2}\left(\frac{\boldsymbol{r}_{ds}\cdot\dot{\boldsymbol{r}}_{ds}}{r}\right)^2 + \boldsymbol{r}_{ds}\cdot\ddot{\boldsymbol{r}}_{ds}\right)\boldsymbol{F}_c \;+ \\ &\quad \frac{1}{c^4}\,(\ldots)\,\boldsymbol{F}_c \;+\; O(\|\dot{\boldsymbol{r}}_{ds}\|)^3. \end{aligned} \tag{23}$$

Because the term with the factor $1/c^4$ can be neglected, this corresponds exactly to the classical Weber formula, which in vector notation reads

$$\boldsymbol{F}_{ds} = \left(1 - \frac{3}{2}\left(\frac{\boldsymbol{r}_{ds}\cdot\dot{\boldsymbol{r}}_{ds}}{\|\boldsymbol{r}_{ds}\|\, c}\right)^2 + \frac{\|\dot{\boldsymbol{r}}_{ds}\|^2}{c^2} + \frac{\boldsymbol{r}_{ds}\cdot\ddot{\boldsymbol{r}}_{ds}}{c^2}\right)\boldsymbol{F}_c. \tag{24}$$

The previous derivations show that a potential energy (19) can be defined for which, together with the kinetic energy (11) and

Newton's second law (13), the law of conservation of energy is satisfied. Furthermore, the potential energy (19) is clearly compatible with classical Weber electrodynamics and the wave equation (1) obtained from Maxwell's equations. The latter is not the case for alternative definitions of the potential energy in Weber electrodynamics [21]. Moreover, the potential energy (19), like the force (21), is recognized to be a purely relative quantity that does not depend on the velocity of a uniformly moving observer.

B. Conservation of energy

We now prove that in Weber-Maxwell electrodynamics, the total energy is a conserved quantity even if the system consists of n particles, where n may be arbitrarily large. To provide a demonstration, we write the total energy E of the system as a sum of all kinetic and potential energies:

$$E = \sum_{k=1}^{n} T_k + \sum_{k=1}^{n} \sum_{i=k+1}^{n} U_{ki}. \tag{25}$$

The n equations of motion (13) are for $k = 1, \ldots, n$

$$\sum_{i=1}^{n} \boldsymbol{F}_{ki} = m_k \ddot{\boldsymbol{r}}_k. \tag{26}$$

We define the force $\boldsymbol{F}_{ii}$ of the i-th particle on itself as zero in order to avoid handling exceptions in the summation. Multiplication by $\dot{\boldsymbol{r}}_k$ yields

$$\sum_{i=1}^{n} \boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_k = m_k \ddot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k. \tag{27}$$

The time derivative of the kinetic energy of the k-th particle is due to equation (11)

$$\dot{T}_k = m_k \ddot{\boldsymbol{r}}_k \cdot \boldsymbol{r}_k, \tag{28}$$

i.e., equation (27) can be rewritten as

$$\sum_{i=1}^{n} \boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_k = \dot{T}_k. \tag{29}$$

The sum of all n equations of motion gives

$$\sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_k = \sum_{k=1}^{n} \dot{T}_k. \tag{30}$$

By formally manipulating the summation indices, we obtain

$$\frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ik} \cdot \dot{\boldsymbol{r}}_i = \sum_{k=1}^{n} \dot{T}_k. \tag{31}$$

Because of Newton's third law (9), $\boldsymbol{F}_{ik} = -\boldsymbol{F}_{ki}$ applies. By using this and the relation $\dot{\boldsymbol{r}}_{ki} = \dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_i$, equation (31) can be simplified to

$$\frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_{ki} = \sum_{k=1}^{n} \dot{T}_k. \tag{32}$$

But because of equation (17), it follows that

$$\boldsymbol{F}_{ki} \cdot \dot{\boldsymbol{r}}_{ki} = -\dot{U}_{ki}. \tag{33}$$

This allows formula (32) to be written as

$$\sum_{k=1}^{n} \dot{T}_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \dot{U}_{ki} = 0. \tag{34}$$

However, because of the symmetry of the potential energy $U_{ki} = U_{ik}$, the equation

$$\frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \dot{U}_{ki} = \sum_{k=1}^{n} \sum_{i=k+1}^{n} \dot{U}_{ki} \tag{35}$$

holds, and it follows that

$$\sum_{k=1}^{n} \dot{T}_k + \sum_{k=1}^{n} \sum_{i=k+1}^{n} \dot{U}_{ki} = 0. \tag{36}$$

However, this corresponds to the time derivative of the total energy, as can be recognized by comparison with equation (25). Clearly, $\dot{E} = 0$ applies, thus indicating that the total energy E is a conserved quantity.

C. Conservation of momentum

For an isolated system of n point charges, the temporal change in the momentum $\boldsymbol{p}_k$ of the k-th point charge is equal to the sum of all forces acting on this point particle. Therefore

$$\dot{\boldsymbol{p}}_k = \sum_{i=1}^{n} \boldsymbol{F}_{ki}, \tag{37}$$

provided that $\boldsymbol{F}_{ii} := 0$ is defined. The temporal change in the total momentum $\boldsymbol{p}$ of the system is the sum of the temporal momentum changes of all point particles. For this reason,

$$\dot{\boldsymbol{p}} = \sum_{k=1}^{n} \dot{\boldsymbol{p}}_k = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki}. \tag{38}$$

Again, Newton's third law (9) is ensured to be valid under all circumstances, and it follows that:

$$\dot{\boldsymbol{p}} = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki} = -\sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ik}. \tag{39}$$

By formally renaming the summation indices, the following is then obtained:

$$\dot{\boldsymbol{p}} = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki} = -\sum_{i=1}^{n} \sum_{k=1}^{n} \boldsymbol{F}_{ki} = -\sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{F}_{ki}, \tag{40}$$

which can be true only if $\dot{\boldsymbol{p}}$ is exactly zero. Therefore, the total momentum $\boldsymbol{p}$ must be a conserved quantity.

Furthermore, the sum of all forces acting on the matter particles disappears in an isolated system. Consequently, electromagnetic waves themselves do not have any momentum. Instead, momentum is a property that only particles of matter can possess. The electromagnetic force, in contrast, is only the mediator. Nevertheless, in a subsystem of an electromagnetic wave and a receiver of the force without the source, the electromagnetic wave seems to possess momentum, because it causes a momentum change at the receiver. However, one must not forget that in another location, the source of the wave also simultaneously experiences a compensating momentum change. Electromagnetic waves therefore have momentum only in non-isolated systems.

D. Conservation of angular momentum

The conservation of angular momentum $\boldsymbol{L}$ can also be demonstrated. The total angular momentum of a system of n point charges is defined by

$$\boldsymbol{L} := \sum_{k=1}^{n} \boldsymbol{r}_k \times \boldsymbol{p}_k. \tag{41}$$

The derivative with respect to time yields

$$\dot{\boldsymbol{L}} = \sum_{k=1}^{n} \dot{\boldsymbol{r}}_k \times \boldsymbol{p}_k + \sum_{k=1}^{n} \boldsymbol{r}_k \times \dot{\boldsymbol{p}}_k. \tag{42}$$

Because the momentum $\boldsymbol{p}_k$ is always parallel to the velocity $\dot{\boldsymbol{r}}_k$, it follows that $\dot{\boldsymbol{r}}_k \times \boldsymbol{p}_k = \boldsymbol{0}$. Thus, equation (42) becomes

$$\dot{\boldsymbol{L}} = \sum_{k=1}^{n} \boldsymbol{r}_k \times \dot{\boldsymbol{p}}_k. \tag{43}$$

Substituting equation (37) gives

$$\dot{\boldsymbol{L}} = \sum_{k=1}^{n} \boldsymbol{r}_k \times \sum_{i=1}^{n} \boldsymbol{F}_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_k \times \boldsymbol{F}_{ki}. \tag{44}$$

This can be further rewritten as

$$\begin{aligned} \dot{\boldsymbol{L}} &= \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_k \times \boldsymbol{F}_{ki} + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_k \times \boldsymbol{F}_{ki} \\ &= \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_k \times \boldsymbol{F}_{ki} + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ik}. \end{aligned} \tag{45}$$

Because of Newton's third law (9), it follows that

$$\dot{\boldsymbol{L}} = \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_k \times \boldsymbol{F}_{ki} - \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ki}, \tag{46}$$

i.e.,

$$\dot{\boldsymbol{L}} = \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{r}_{ki} \times \boldsymbol{F}_{ki} \tag{47}$$

applies. However, the Weber force (6) is a central force, and $\boldsymbol{r}_{ki} \times \boldsymbol{F}_{ki} = \boldsymbol{0}$ holds. Thus, we obtain

$$\dot{\boldsymbol{L}} = \boldsymbol{0}. \tag{48}$$

This shows that the total angular momentum of an isolated system must also be a conserved quantity.

E. Necessary features of valid dynamics

The previous sections show which features are needed for which steps in the proofs of the conservation laws. Of note, Newton's second law and the kinetic energy could also be defined such that they would converge to the classical versions for small relative velocities, on the one hand, and would become compatible with the empirical results of high energy physics, on the other hand. We must only ensure that the presented proofs retain their validity. Therefore, certain conditions must be met, which should therefore be identified and summarized.

The following features are central to the proofs of the conservation laws:

1) Newton's third law (9): this property is necessary for the proof of all three conservation laws.
2) The time derivative of the kinetic energy must be equal to the scalar product of time derivative of momentum and velocity: this property is needed in the step from equation (12) to equation (14), i.e., to prove the conservation of energy.
3) The momentum must be parallel to the velocity: this is required in the proof of conservation of angular momentum.
4) The force formula between two point charges must be a central force: this is also needed in the proof of conservation of angular momentum.

As becomes clear, the last two features are needed only in the proof of the conservation of angular momentum. The property 4) in Weber electrodynamics is satisfied only if the two point charges are so close to each other that the finite propagation speed of the force is unimportant. The conservation of angular momentum is therefore the weakest of the three conservation laws. Conservation of momentum, in contrast, is always guaranteed in every case in Weber-Maxwell electrodynamics. Somewhat less clear is the conservation of energy, because the possibility of defining a potential energy is required.

Of note, a fourth conserved quantity exists in Newtonian mechanics, which is referred to as the Runge-Lenz vector. This conservation law is satisfied for only very simple force laws, such as Coulomb's law or Newton's law of universal gravitation. In fact, measurements show that the Runge-Lenz vector is often not a conserved quantity in real physical systems [22].

IV. Magnetism in Weber-Maxwell electrodynamics

Because this article is also intended for readers unfamiliar with Weber electrodynamics, a brief discussion of the question of why the concept of magnetism is obsolete in Weber electrodynamics follows. We begin with a simple example and consider an infinitely long straight wire on the x-axis. Let an electric current I flow in this wire. Furthermore, somewhere on the z-axis at z, let there be a point-like test charge q moving with velocity $\boldsymbol{v} = v_x\, \boldsymbol{e}_x + v_y\, \boldsymbol{e}_y + v_z\, \boldsymbol{e}_z$.

In standard electrodynamics, an infinitely long wire in which a current I flows is surrounded by a magnetic field $\boldsymbol{B} = -\frac{\mu_0}{2\pi z}\, I\, \boldsymbol{e}_y$. The force on the test charge q is obtained by using the Lorentz force $\boldsymbol{F} = q\,\boldsymbol{E} + q\,\boldsymbol{v} \times \boldsymbol{B}$. Because the wire is electrically neutral, and therefore the electric field $\boldsymbol{E}$ vanishes, the force is

$$\boldsymbol{F} = -\frac{\mu_0\, q\, I}{2\,\pi\, z}\, \boldsymbol{v} \times \boldsymbol{e}_y. \tag{49}$$

The same result can be obtained with Weber electrodynamics. To provide a demonstration, we model the current I with resting metal ions $+e$ and an equal number of moving electrons of charge $-e$, which move with an average drift velocity of $\boldsymbol{u} = -u\, \boldsymbol{e}_x$. Because the electrons drift to the left, the technical direction of the current points to the right. The intensity of the resulting current is $I = e\,n\,u$, where n represents the number of moving electrons per meter of the conductor length.

In contrast to Lorentz-Einstein electrodynamics, magnetism in Weber electrodynamics is not a fundamental force but is a multi-particle effect, i.e., the net force of all individual forces between the test charge q and the charge carriers inside the wire. Expressed as an integral, the resulting force is

$$\begin{aligned} \boldsymbol{F} \;=\; & \int_{-\infty}^{+\infty} \boldsymbol{F}_W(q, n\,(-e), z\,\boldsymbol{e}_z - s\,\boldsymbol{e}_x, \boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}s \;+ \\ & \int_{-\infty}^{+\infty} \boldsymbol{F}_W(q, n\,(+e), z\,\boldsymbol{e}_z - s\,\boldsymbol{e}_x, \boldsymbol{v})\,\mathrm{d}s, \end{aligned} \tag{50}$$

where $\boldsymbol{F}_W$ is the Weber force (6) in modern representation. This integral can be solved exactly, and the following approximation can be obtained

$$\boldsymbol{F} \approx -\frac{c^4\,\sqrt{c^2 - v^2 + v_x^2}}{(c^2 - v^2)^{3/2}\,(c^2 - v_y^2)}\;\frac{q\,e\,n\,u}{2\,\pi\,c^2\,\varepsilon_0\,z}\,\boldsymbol{v} \times \boldsymbol{e}_y \tag{51}$$

for very small electron drift velocities $u \ll c$. This in turn becomes identical to equation (49) when $e\,n\,u = I$ and $c^2\,\varepsilon_0 = 1/\mu_0$ is used, provided that equation (51) regarding $\|\boldsymbol{v}\| := v$ is expressed as a Taylor series of second order. Of note, the Fechner hypothesis is not necessary [2, p. 87-88] and moreover is irrelevant for the result with which velocity one moves relative to the wire and to the test charge. The later aspect is not the case in Lorentz-Einstein electrodynamics.

The simple example above illustrates that magnetism is not a fundamental force from the viewpoint of Weber electrodynamics. In fact, the concept of the magnetic field is obsolete not only in the case of an infinitely long wire but in general. This represents a significant conceptual simplification, even though solving integrals of the type (50) may seem somewhat tedious.

However, this integration method allows every practical problem to be solved numerically. This aspect is also true for high-frequency alternating current, because the solution of the wave equation (1) can be used for the point antenna[3]:

$$\begin{aligned} \boldsymbol{F} \;=\; & \frac{q_d\,q\,\gamma(\boldsymbol{v})\left(1 - \frac{\boldsymbol{v}}{c}\cdot\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}\right)}{2\,\pi\,\varepsilon_0\,c^2\,\|\boldsymbol{r}\|}\Bigg\{ \\ & \left(\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \times \left(\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \times \ddot{\boldsymbol{s}}\,(t+\tau)\right)\right) - \\ & \left(\left(\frac{\boldsymbol{v}}{c} \times \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}\right) \times \ddot{\boldsymbol{s}}\,(t+\tau)\right)\Bigg\}. \end{aligned} \tag{52}$$

In equation (52), $\boldsymbol{s}(t)$ is the time-dependent infinitesimal displacement of the charges $+q$ and $-q$ in a bound particle located at the coordinate origin. The time constant τ is given by

$$\tau = \frac{\boldsymbol{r}\cdot\boldsymbol{v} - \sqrt{c^2\,r^2 - \|\boldsymbol{r}\times\boldsymbol{v}\|^2}}{c^2 - v^2} \tag{53}$$

[3]The derivation of the wave equation (1) from Maxwell's equations and finding the solution for a point antenna is described in [19]. Only the far-field approximation is given here, and an adjustment of the sign of the velocity $\boldsymbol{v}$ was performed.

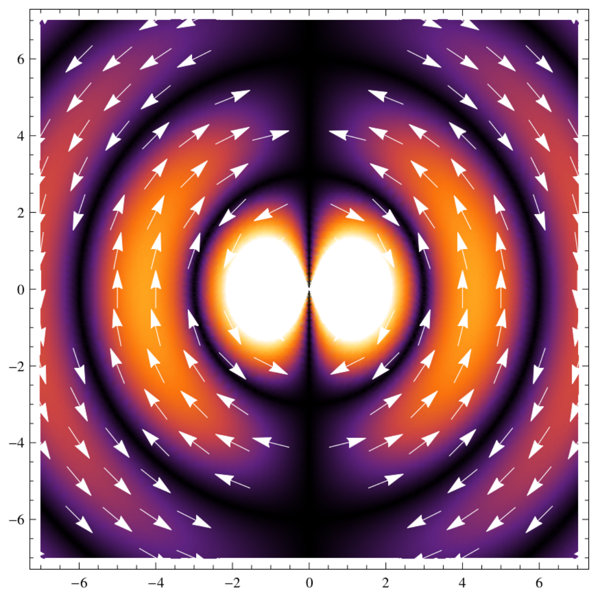

Fig. 3. The figure illustrates the field of the electromagnetic force of a point antenna oscillating with a frequency of 50 MHz in the z-direction. The field is shown as would be perceived by a test charge at rest in the x-z plane (in m). The darker the background, the smaller the magnitude of the field strength in N.

thereby ensuring that the electromagnetic wave travels at speed of light c for a test charge q_d moving with relative velocity $\boldsymbol{v}$ at location $\boldsymbol{r}$.

Figure 3 shows the field of the force of a point antenna polarized in the z-direction at the coordinate origin, from the perspective of a test charge q_d resting relative to the point antenna somewhere in the x-z plane. As can be seen, the field of the force corresponds exactly to the electric field of the Hertzian dipole, as indicated in many textbooks.

The situation is different when the test charge is moving. Figure 4 shows an example of the same field from the perspective of a point charge moving with speed $c/3$ to the right. As clearly indicated, a Doppler effect now occurs. If the test charge is located at the left side of the point antenna, a blueshift occurs, because the test charge moves toward the point antenna. On the right side of the point antenna, in contrast, a redshift is seen.

For basic understanding, an important and critical aspect is that this field is real only for a moving point charge. A stationary observer would perceive the same field as in Figure 3. However, the observer could measure the acceleration of the test charge and indirectly infer the field that is actually perceived by a moving test charge, which would correspond exactly to the field in Figure 4 and not to the field measured by the observer.

The figures 3 and 4 show the electromagnetic forces on a test charge in its entirety; consequently, the effects that would resemble the Lorentz force are difficult to recognize. However, the magnetic effects are also very real in Weber-Maxwell electrodynamics, even though the concept of the magnetic field

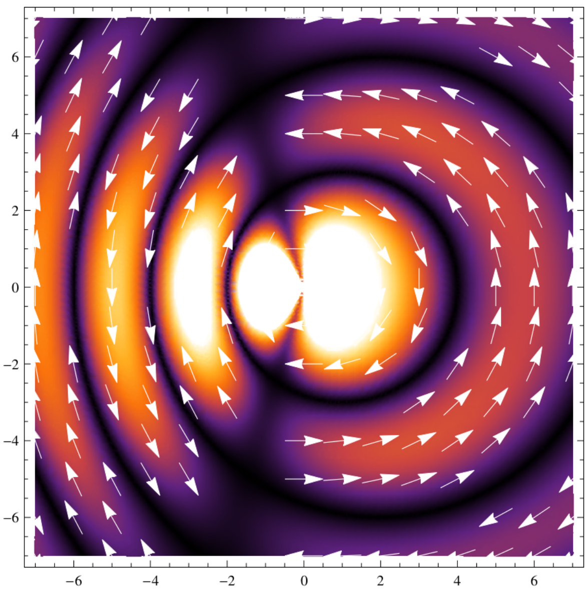

Fig. 4. The same situation as in Figure 3. Here, however, the test charge moves to the right with speed $c/3$.

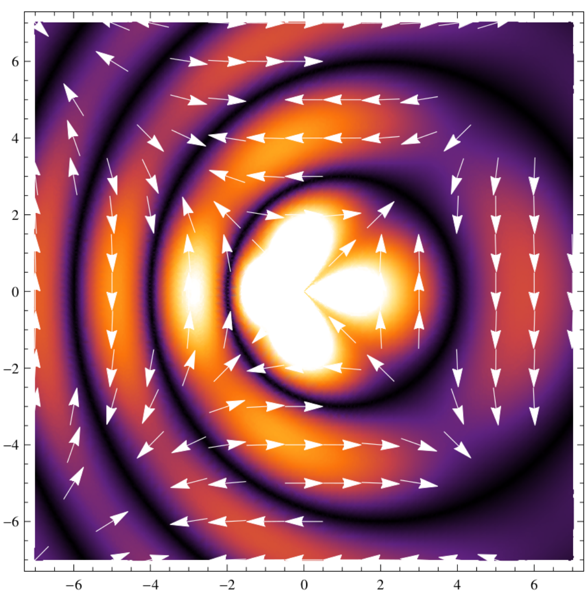

Fig. 5. Only the velocity-dependent force components are shown. The test charge moves to the right with speed $c/3$. In the x-y plane; i.e., transverse to the axis of oscillation, the force acts perpendicularly to the direction of motion.

is no longer needed. To illustrate this aspect, Figures 5 and 6 show the force components that remain when all components of the electromagnetic force that do not depend on the velocity are subtracted[4]. In this way, the pendant of the magnetic force field in Weber-Maxwell electrodynamics is obtained, although the comparison must not be taken too literally.

Figure 5 shows this velocity-dependent force component for a test charge moving to the right. As can be seen, the force

[4]Set $\boldsymbol{v}$ to zero everywhere except in τ and subtracte this from the force.

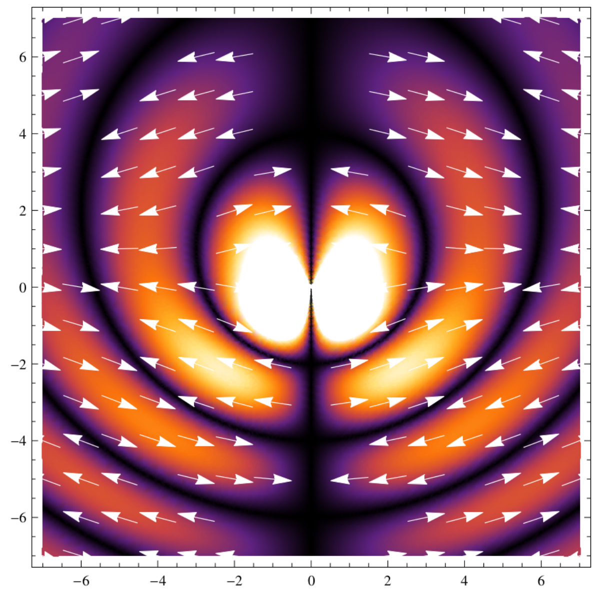

Fig. 6. Only the velocity-dependent force components are shown. The test charge moves upward with speed $c/3$. In the x-y plane; i.e., transverse to the axis of oscillation, the force acts perpendicularly to the direction of motion.

component in the x-y plane agrees exactly with the expectation according to Lorentz-Einstein electrodynamics, because the moving test charge experiences a force that is oriented exactly perpendicular to the direction of motion. The same is true for Figure 6. Here, the test charge is moving in the z-direction. As can be seen, the force also acts in the x-y plane orthogonally to the direction of motion of the test charge.

However, the equivalence to the Lorentz force is present in only the x-y plane, i.e., transverse to the axis of oscillation. Above and below, in contrast, the predictions of Weber-Maxwell electrodynamics and Lorentz-Einstein electrodynamics are partly diametrically different. However, these differences often do not matter in practice, because an alternating current carrying wire can be interpreted as a chain of many point antennas in the direction of oscillation. Therefore, in the examples in Figures (3) to (6), the fields of additional point antennas must be imagined on the z-axis. The fields of all point antennas superimpose, and the force components typical of Weber electrodynamics are cancelled out because of the symmetry.

This symmetry is the reason why Lorentz-Einstein electrodynamics and Weber-Maxwell electrodynamics are de facto equivalent for closed conductor loops. However, in many applications in electrical engineering, this equivalence is not the case. For example, a capacitor represents a gap in a conductor loop. Indeed, forces between the plates can be calculated with Lorentz-Einstein electrodynamics, which cannot be real, because they violate the conservation laws. However, if the force effects are calculated with Weber-Maxwell electrodynamics, realistic forces are always obtained that satisfy the conservation laws and agree perfectly with the measurement results in reality.

V. Summary and conclusion

The article has shown that the modern formulation of Weber electrodynamics by means of the wave equation (1) is a theory of electromagnetism that, in the nonrelativistic domain, is clearly superior to standard electrodynamics, because the concept of the magnetic field becomes completely obsolete, and the conservation laws are ensured to hold even in the presence of electromagnetic waves. Furthermore, Weber-Maxwell electrodynamics is also considerably more compact and decreases the necessary number of fundamental equations to a single equation.

The last field of application in which Lorentz-Einstein electrodynamics seems necessary is when electric charges move with velocities close to c. This scenario is usually the case only in designing particle accelerators or nuclear fusion research. How Weber-Maxwell electrodynamics can be applied in such extreme applications is currently an open question and a subject of current research.

Importantly, in the field of electrical engineering, Weber-Maxwell electrodynamics allows many types of problems to be rapidly and easily solved; applications include electrical power engineering, radio-frequency engineering, plasma physics, electronics and computational electromagnetics. In particular, the basic solution for a uniformly moving point charge in the case of direct current, or a point antenna in the case of alternating current, can be numerically integrated along the current paths. Therefore, the need to solve Maxwell's equations, which not only requires experience in dealing with partial differential equations, but also can lead to paradoxical and unrealistic results, is eliminated. However, Weber-Maxwell electrodynamics ensures that the solutions always satisfy the conservation laws and are consequently reasonable and realistic.

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